THE PROBLEM OF ELECTRIC CURRENT EDDIES AT ENTRY OF AN ANISOTROPICALLY CONDUCTING MEDIUM TO A MAGNETIC FIELD

PMM Vol. 35, №2, 1971, pp. 229-237

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(Received October 12, 1970)

The electric field in a stream of medium with tensor conductivity dependent on the Hall effect is analyzed in the region of abrupt change in the external magnetic field. It is shown that the pattern of electric current eddies arising in this region, as well as the electromagnetic forces and the Joule dissipation of induced currents are essentially defined by the velocity fields and the physical properties of the moving medium, whose parameters (electrical conductance and the Hall effect) in a varying magnetic field cannot be generally considered as constant.

From the mathematical point of view the solution of the input Riemann-Hilbert boundary value problem reduces by means of analytic extension on the principle of symmetry to solving the inhomogeneous Riemann problem with discontinuous coefficients.

The problem of motion of a conducting medium in the inhomogeneity region of an external magnetic field naturally occurs in investigations of phenomena in the end zones of magnetohydrodynamic channels at the in- and outlet of the plasma stream in and out of the magnetic field. Sutton [1] considered this problem on the assumption of constancy of the medium velocity and physical properties. His solution was later used in [2] for determining distortions of the velocity profile.

1. Let an ionized gas in which anisotropy of conductance is produced by the Hall effect move $\mathbf{v}\,(u\,(x,\ y),\ v\,(x,\ y),\ 0)$ in a plane channel $0 < y < h,\ -\infty < x < \infty$ with nonconducting walls, and let the external magnetic field $\mathbf{H}\,(0,\ 0,\ H_z(x))$ normal to the direction of flow of gas vary step-wise from $H_1\ (x < 0)$ to $H_2\ (x > 0)$ for $x = 0,\ 0 < y < h$, while remaining constant in each of the half-strips $x < 0$ and $x > 0$.

In a theoretical investigation it is expedient to satisfy an inhomogeneous magnetic field by a step-wise function as the idealization of a magnetic field with a high gradient. This permits to examine the effects in conditions of sharply defined electric current eddies. It also makes possible the derivation of a complete analytic solution for an arbitrary velocity field by the effective methods of the theory of boundary value problems. This assumption does not introduce any appreciable error into the results, provided the length l along which the magnetic field varies is small (e. g. in comparison with the channel width $l \ll h$). The finite value of l makes the current eddies less clearly defined and the integral magnetic forces and the Joule dissipation of induced current smaller than those in the limit case of $l \to 0$.

The following two assumptions are usually made in this class of problems: owing to the smallness of the magnetic Reynolds number $(R_m \ll 1)$, the induced magnetic current field in plasma is neglected and the plasma itself is considered to be an incompressible medium.

In accordance with these assumptions the input system of equations can for the isothermic case be written as

$$E_x(x, y) = \frac{1}{\sigma(H)} j_x(x, y) + R(H) H j_y(x, y) - \frac{1}{c} vH$$
$$E_y(x, y) = -\frac{1}{\sigma(H)} j_y(x, y) - R(H) H j_x(x, y) + \frac{1}{c} uH \qquad (1.1)$$
$$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{j} = 0, \quad \nabla \cdot \mathbf{v} = 0$$

where the electrical conduction $\sigma(H)$ and the Hall effect $R(H)$, dependent on the magnetic field, are determined in terms of components $\sigma_{xx}(H)$ and $\sigma_{xy}(H)$ of electrical conduction tensors are expressed by

$$\sigma(H) = \frac{\sigma_{xx}(H) + \sigma_{xy}^2(H)}{\sigma_{xx}(H)}, \qquad R(H) = \frac{\sigma_{xy}(H)}{H[\sigma_{xx}^2(H) + \sigma_{xy}^2(H)]} \qquad (1.2)$$

It follows from the system of equations (1.1) that in each of the half-strips ($0 < y < h$, $x < 0$ and $x > 0$), where $H_k = \text{const}$ ($k = 1, 2$), we can introduce the complex electric current

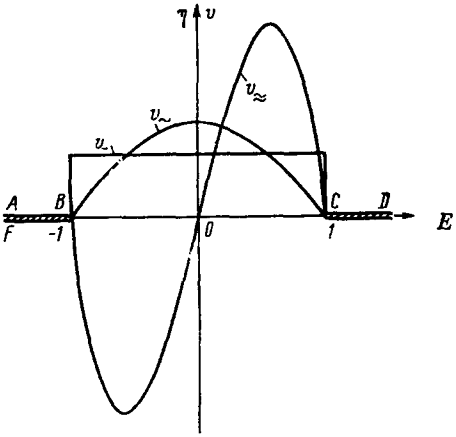

Fig. 1

$$j(z) = j_x(x, y) - ij_y(x, y) \quad (z = x + iy)$$

We map strip $-\infty < x < \infty, 0 < y < h$ in plane $\zeta = \xi + i\eta$ using function $\zeta = \mathrm{ch}(\pi z/h)$. The half-strips $0 < y < h$, $x > 0$ and $x < 0$ are then represented in the upper and lower half-planes, and sections along rays $\eta = 0$, $\xi < -1$ and $\xi > 1$ correspond to boundary lines $x = 0$ and $x = h$.

By stating the boundary value problem in the transformation region we obtain an exact solution of the problem of flow of an anisotropically conducting medium through the combined cross section $\eta = 0$, $-1 < \xi < 1$ of two "channels" represented by the two half-planes $\mathrm{Im}\,\zeta > 0$ and $\mathrm{Im}\,\zeta < 0$ in which $H_k = \text{const}$, while in the separating plane itself it changes abruptly ($H_1 \neq H_2$). The solution for the strip is obtained by inverse transformation $\zeta \to z$ of the solution derived below.

Distribution of the field current is found by solving system (1.1) with the following boundary conditions: at the cross section of the plasma flow ($\eta = 0$, $-1 < \xi < 1$) the normal component of the current density vector and the tangent component of the field tension are continuous, while along the remaining sections of the ξ-axis (dielectrics) $j_{\eta k} = 0$, $k = 1,2$ (in the following subscripts 1 and 2 denote functions in the half-planes $\mathrm{Im}\,\zeta > 0$ and $\mathrm{Im}\,\zeta < 0$, respectively, Fig. 1)

$$\begin{array}{ll} j_{\eta 1} = 0 \ \text{ on } AB \text{ and } CD, & \\ j_{\eta 2} = 0 \ \text{ on } FB \text{ and } CE, & j_{\eta 1} = j_{\eta 2}, \quad E_{\xi 1} = E_{\xi 2} \ \text{ on } BC \end{array} \qquad (1.3)$$

Since current eddies exist only in the region adjacent to the magnetic field inhomogeneity, the current vanishes at infinity. To refine the asymptotic conditions at $|\zeta| \to \infty$, which is basic to this statement of the boundary value problem, it is necessary to take into consideration the obvious integral relationship

$$\int_{-1}^{1} j_{\eta k}(\xi, 0)\, d\xi = 0 \tag{1.4}$$

valid in the absence of any external sources of current in the ζ-plane.

2. Taking advantage of the problem symmetry, we introduce into our considerations two piece-wise holomorphic functions

$$\begin{aligned} \Psi_1(\zeta) &= \begin{cases} \Psi_1^+(\zeta) = j_1(\zeta) = j_{\xi 1}(\xi, \eta) - i j_{\eta 1}(\xi, \eta) & \text{for } \operatorname{Im} \zeta > 0 \\ \Psi_1^-(\zeta) = \bar{j}_1(\zeta) = j_{\xi 1}(\xi, -\eta) + i j_{\eta 1}(\xi, -\eta) & \text{for } \operatorname{Im} \zeta < 0 \end{cases} \\ \Psi_2(\zeta) &= \begin{cases} \Psi_2^+(\zeta) = \bar{j}_2(\zeta) = j_{\xi 2}(\xi, \eta) + i j_{\eta 2}(\xi, \eta) & \text{for } \operatorname{Im} \zeta > 0 \\ \Psi_2^-(\zeta) = j_2(\zeta) = j_{\xi 2}(\xi, -\eta) - i j_{\eta 2}(\xi, -\eta) & \text{for } \operatorname{Im} \zeta < 0 \end{cases} \end{aligned} \tag{2.1}$$

According to the boundary conditions (1.3) functions $\Psi_k(\zeta)$ $(k = 1, 2)$ satisfy relationships

$$\Psi_1^+(\zeta) = -\Psi_2^+(\zeta), \qquad \Psi_1^-(\zeta) = -\Psi_2^-(\zeta) \tag{2.2}$$

and can be represented by the integral of the Cauchy kind

$$\Psi_k(\zeta) = (-1)^k \frac{1}{\pi} \int_{-1}^{1} \frac{\gamma(\xi)\, d\xi}{\xi - \zeta} \quad (k = 1, 2) \tag{2.3}$$

where $\gamma(\xi) = j_{\eta k}(\xi)$. The series expansion of (2.3) in decreasing powers of ζ with relationship (1.4) taken into account yields at considerable $|\zeta|$ for $\Psi_k(\zeta)$ the following expression:

$$\Psi_k(\zeta) = (-1)^{k-1}\left[\frac{A}{\zeta^2} + O\left(\frac{1}{\zeta^3}\right)\right], \qquad A = \frac{1}{\pi} \int_{-1}^{1} \xi \gamma(\xi)\, d\xi \tag{2.4}$$

The boundary conditions rewritten for the complex current parameters, for example, in the upper half-plane, become now of the form

$$\begin{gathered} (\sigma_1 + \sigma_2)\, j_{\xi 1} + (\sigma_2 \beta_1 - \sigma_1 \beta_2)\, j_{\eta 1} = \frac{\sigma_1 \sigma_2}{c} v (H_1 - H_2) \quad \text{along } L_1 \quad (L_1 = BC) \\ j_{\eta 1} = 0 \ \text{ along } L_2 \quad (L_2 = AB + CD = FB + CE) \end{gathered} \tag{2.5}$$

where

$$\beta_k = \sigma_k R_k H_k = \sigma(H_k) R(H_k) H_k \quad (k = 1, 2) \tag{2.6}$$

Using the piece-wise holomorphic functions $\Psi_k(\zeta)$ defined in (2.1), we reduce the Riemann-Hilbert problem (2.5) to the following nonhomogeneous Riemann problem with discontinuous coefficients:

$$\begin{gathered} \Psi_1^+(\xi) = -\frac{\sigma_1 + \sigma_2 - i(\sigma_2\beta_1 - \sigma_1\beta_2)}{\sigma_1 + \sigma_2 + i(\sigma_2\beta_1 - \sigma_1\beta_2)} \Psi_1^-(\xi) + \frac{2\sigma_1\sigma_2 (H_1 - H_2)\, v(\xi)}{c[\sigma_1 + \sigma_2 + i(\sigma_2\beta_1 - \sigma_1\beta_2)]} \ \text{along } L_1 \\ \Psi_1^+(\xi) = \Psi_1^-(\xi) \quad \text{along} \quad L_2, \qquad \Psi_1(\zeta) = \frac{A}{\zeta^2} + O(\zeta^{-3}) \end{gathered} \tag{2.7}$$

The general solution of the problem (2.7) is provided by formula [3, 4]

$$\Psi_1(\zeta) = \frac{\sigma_1\sigma_2 (H_1 - H_2)\, X_\nu(\zeta)}{\pi c i\, [\sigma_1 + \sigma_2 + i(\sigma_2\beta_1 - \sigma_1\beta_2)]} \int_{-1}^{1} \frac{v(\xi)\, d\xi}{X_\nu^+(\xi)(\xi - \zeta)} \quad (-1 < \xi < 1) \tag{2.8}$$

here $v(\xi)$ is a given function at L_1 which determines the velocity profile, and must satisfy the Hölder condition. The form of the canonical function $X_\nu(\zeta)$ is established by the choice of the velocity profile along the line of the magnetic field discontinuity.

By varying $v(\xi)$ the complete set of solutions admitted by the Riemann problem (2.7) and comprised in (2.8) can be divided into classes depending on the fulfilment of certain integral conditions satisfied by function $v(\xi)$ along L_1 or, in the terminology of boundary value problems, on the Riemann problem index. Omitting intermediate computations, we present the final result:

1) The solution bounded in the vicinity of wall ABF and unbounded in the vicinity of wall DCE (solution with zero index $\varkappa = 0$). In this case the condition imposed on $v(\xi)$, and the form of the canonical function $X_B(\zeta)$ are defined by formulas

$$\int_{-1}^{1} v(\xi)(1+\xi)^{-1/2-\varepsilon}(1-\xi)^{1/2+\varepsilon}\,d\xi = 0, \quad X_B(\zeta) = (\zeta+1)^{1/2+\varepsilon}(\zeta-1)^{-1/2-\varepsilon} \tag{2.9}$$

$$\varepsilon = \frac{1}{\pi}\operatorname{arc\,tg}\frac{\sigma_2\beta_1 - \sigma_1\beta_2}{\sigma_1+\sigma_2}, \qquad 0 \leqslant |\varepsilon| < \frac{1}{2} \tag{2.10}$$

Here and in subsequent formulas the canonical function $X_\nu(\zeta)$ is understood to be that branch which is holomorphic in the ζ-plane and has positive real values $\operatorname{Re} X_\nu^+(\xi)$ on the upper side of L_1.

2) The solution bounded in the vicinity of wall DCE and unbounded in the vicinity of wall ABF ($\varkappa = 0$)

$$\int_{-1}^{1} v(\xi)(1+\xi)^{1/2-\varepsilon}(1-\xi)^{-1/2+\varepsilon}\,d\xi = 0, \quad X_C(\zeta) = (\zeta+1)^{-1/2+\varepsilon}(\zeta-1)^{1/2-\varepsilon} \tag{2.11}$$

3) The solution bounded in the vicinity of the two walls ABF and DCE (negative index $\varkappa = -1$). The law of variation of $v(\xi)$ satisfies in this case the integral relationships

$$\int_{-1}^{1} v(\xi)(1+\xi)^{-1/2-\varepsilon}(1-\xi)^{-1/2+\varepsilon}\,d\xi = 0$$

$$\int_{-1}^{1} v(\xi)\,\xi\,(1+\xi)^{-1/2-\varepsilon}(1-\xi)^{-1/2+\varepsilon}\,d\xi = 0 \tag{2.12}$$

the first of which is clearly the sum of integral conditions (2.9) and (2.11).

The canonical function is given by formula

$$X_{AB}(\xi) = (\zeta+1)^{1/2+\varepsilon}(\zeta-1)^{1/2-\varepsilon} \tag{2.13}$$

4) The solution unbounded in the vicinity of the two walls ABF and DCE ($\varkappa = 1$). If $v(\xi)$ does not satisfy along L_1 any of the conditions defined by (2.9) and (2.11) or to system (2.12), then in formula (2.8) we have

$$X_\nu(\zeta) = X_0(\zeta), \qquad X_0(\zeta) = (\zeta+1)^{-1/2+\varepsilon}(\zeta-1)^{-1/2-\varepsilon} \tag{2.14}$$

3. Let us now apply the general theory of field calculation to the problem of entry of a conducting medium into a magnetic field for three characteristic velocity profiles (Fig. 1):

$$v_{-} = V = \text{const}, \quad v_{\sim} = \frac{3}{2}V(1-\xi^2), \quad v_{\approx} = 8V\xi(1-\xi^2) \tag{3.1}$$

The choice of this kind of relationships, generally calculated on the basis of viscous, perfect, and other properties of the moving medium, is dictated by considerations of the analysis of current eddies behavior at varying velocity patterns, when the plasma flow rate Q through the entire channel cross section at v_- and $v_\sim$ and through the channel half-sections at $v_\approx$ is the same

$$Q_- = Q_\sim = Q_\approx = \int_{-1}^{1} v_-(\xi)\,d\xi = \int_{-1}^{1} v_\sim(\xi)\,d\xi = \int_{0}^{1} v_\approx(\xi)\,d\xi \tag{3.2}$$

In the case of an alternating profile of velocity $v_\approx$ the total flow rate across section BC is zero, and we have in essence a velocity vortex in the inhomogeneity zone of the magnetic field.

A direct test of integral conditions (2.9), (2.11) and (2.12) for the relationships (3.1) defining $v(\xi)$ will readily show that the general solution corresponds to the class of solutions (2.14) with singularities at the channel walls.

Using the Cauchy theorem on residues and the formulas of Sokhotskii-Plemel, we obtain for the complex current $j(\zeta) = j_\xi(\xi, \eta) - ij_\eta(\xi, \eta)$ and for the normal and tangent components of current along the discontinuity line of the magnetic field the following expressions:

a) for $v_- = V = \text{const}$.

$$j_-(\zeta) = \Delta(\sigma_k, H_k, V)\,[1 + f_-(\zeta)\,X_0(\zeta)], \quad f_-(\zeta) = 2\varepsilon - \zeta \tag{3.3}$$
$$j_{\xi-}(\xi) = \Delta(\sigma_k, H_k, V)\,[1 - \sin\pi\varepsilon f_-(\xi)\,X'(\xi)]$$
$$j_{\eta-}(\xi) = \Delta(\sigma_k, H_k, V)\cos\pi\varepsilon f_-(\xi)\,X'(\xi)$$

b) for $v_\sim = {}^3/_2 V(1 - \xi^2)$

$$j_\sim(\zeta) = {}^3/_2\Delta(\sigma_k, H_k, V)\,[1 - \zeta^2 + f_\sim(\zeta)\,X_0(\zeta)]$$
$$j_{\xi\sim}(\xi) = {}^3/_2\Delta(\sigma_k, H_k, V)\,[1 - \xi^2 - \sin\pi\varepsilon f_\sim(\xi)\,X'(\xi)] \tag{3.4}$$
$$j_{\eta\sim}(\xi) = {}^3/_2\Delta(\sigma_k, H_k, V)\cos\pi\varepsilon\, X'(\xi)$$
$$f_\sim(\zeta) = \zeta^3 - 2\varepsilon\zeta^2 + (2\varepsilon^2 - {}^3/_2)\,\zeta - {}^4/_3\,\varepsilon^3 + {}^7/_3\varepsilon$$

c) for $v = 8V\xi(1 - \xi^2)$

$$j_\approx(\zeta) = 8\Delta(\sigma_k, H_k, V)\,[\zeta(1 - \zeta) + f_\approx(\zeta)\,X_0(\zeta)]$$
$$f_\approx(\zeta) = \zeta^4 - 2\varepsilon\zeta^3 + (2\varepsilon^2 - {}^3/_2)\,\zeta^2 + ({}^7/_3\,\varepsilon - {}^4/_3\varepsilon^3)\,\zeta + {}^2/_3\varepsilon^4 - {}^5/_3\varepsilon^2 + {}^3/_8 \tag{3.5}$$
$$j_{\xi\approx}(\xi) = 8\Delta(\sigma_k, H_k, V)\,[\xi(1 - \xi)^2 - \sin\pi\varepsilon f_\approx(\xi)\,X'(\xi)]$$
$$j_{\eta\approx}(\xi) = 8\Delta(\sigma_k, H_k, V)\cos\pi\varepsilon f_\approx(\xi)\,X'(\xi)$$

The following notation is used in formulas (3.3)–(3.5):

$$\Delta(\sigma_k, H_k, V) = \frac{\sigma_1\sigma_2(H_1 - H_2)\,V}{c(\sigma_1 + \sigma_2)}, \quad X'(\xi) = (1 + \xi)^{-1/_2+\varepsilon}(1 - \xi)^{-1/_2-\varepsilon} \tag{3.6}$$

One property common to all solutions independent of the form of function $v(\xi)$ should be noted. If relationship

$$\sigma_2\beta_1 - \sigma_1\beta_2 = 0 \tag{3.7}$$

which in accordance with (2.6) may be written in the form

$$R_1(H)H_1 = R_2(H)H_2 \tag{3.8}$$

is satisfied, then $\varepsilon = 0$ and the current distribution is of the same form as in the case of scalar conduction. The physical explanation of this is that the anisotropy of conduction is determined by the Hall effect whose intensity depends, in turn, on $R(H)$ and H. If relationship (3.8) is satisfied, the Hall effect appears uniformly throughout the plasma flow region and the tensor character of conduction is not reflected in the current distribution. Equality (3.7) or its equivalent equality (3.8) are always present in the analysis of electric fields in media with piece-wise anisotropic conduction [3].

Relative current densities

$$j_\xi^* (\xi) = j_\xi (\xi) / \Delta (\sigma, H, V), \qquad j_\eta^* (\xi) = j_\eta (\xi) / \Delta (\sigma, H, V)$$

along the discontinuity line of the magnetic field $-1 < \xi < 1$, $\eta = 0$ for $v_- (\xi)$, $v_\sim (\xi)$ and $v_\approx (\xi)$ are shown in Figs. 2 - 4 for $\beta = 0, 1, 3$ and 10, calculated by formulas (3.3)–(3.6) on the assumption that

$$H = H_1, \quad H_2 = 0, \quad \sigma = \sigma_1 = \sigma_2 \quad \beta = \beta_1 = \sigma R (H) H$$
$$(\varepsilon = \pi^{-1} \operatorname{arctg} \beta / 2)$$

It is seen from Figs. 2 and 3 that the electric field is characterized by the concentration of current at the channel walls (with integrable singularities at points B and C) and by the disturbance of the symmetry of current eddies relative to the η-axis for $\beta \neq 0$. With increasing Hall parameter $\beta = \sigma R (H) H$ the centers of current eddies (points on the ξ-axis at which $j_\eta^* (\xi) = 0$) move toward the channel wall. For $\beta < 10$ the intensity of currents is higher for nonuniform "viscous" profile of velocity $v_\sim (\xi)$ while for considerable β the current distribution in eddies is approximately the same for both $v_- (\xi)$ and $v_\sim (\xi)$. It is interesting to note that in the case of alternating velocity $v_\approx (\xi)$ two centers of eddy currents are present, one of which moves in the direction of the η-axis (Fig. 4), while the second remains almost stationary relative to changes of β.

4. The integral characteristics – the Joule dissipation and the electromagnetic forces of induced currents – provide a clearer picture of the effects of distortion of the velocity profile and of the Hall parameter β on the generation of circulating currents.

The Joule dissipation Q_j in the channel is usually calculated by formula

$$Q_j = \iint \frac{\mathbf{j}^2}{\sigma} d\xi \, d\eta = \iint \left[\mathbf{j} \cdot \mathbf{E} + \frac{1}{c} \mathbf{j} (\mathbf{v} \times \mathbf{H})\right] d\xi \, d\eta \tag{4.1}$$

where the double integral is taken over the whole of the sectioned plane ζ, and the vector notation

$$\mathbf{j} = \sigma \mathbf{E} - \sigma R (H) \mathbf{j} \times \mathbf{H} \tag{4.2}$$

is used for the generalized Ohm's law.

With the use of the Gauss-Ostrogradskii formula we find that the double integral of the first term in brackets is zero, and that formula (4.1), after transposition of $\mathbf{j}$ and $\mathbf{v}$ in the scalar triple product, reduces to

$$Q_j = -\frac{1}{c} \iint \mathbf{v} (\mathbf{j} \times \mathbf{H}) \, d\xi \, d\eta \tag{4.3}$$

This shows that the Joule dissipation in a channel with throughout dielectric walls is numerically equal to the work of electromagnetic forces (the integrand represents the work of the electromagnetic force along path $\mathbf{v}$).

Taking into account the differential relationship

$$dj_\xi = -\frac{\partial j_\eta}{\partial \eta} d\xi + \frac{\partial j_\eta}{\partial \xi} d\eta = \frac{d j_\eta}{d\eta} d\xi \tag{4.4}$$

which follows from equations

$$\frac{\partial j_\xi}{\partial \xi} + \frac{\partial j_\eta}{\partial \eta} = 0, \qquad \frac{\partial j_\eta}{\partial \xi} - \frac{\partial j_\xi}{\partial \eta} = 0 \tag{4.5}$$

and, also, the boundary values of $\mathbf{j}$ and $\mathbf{v}$ along $L = L_1 + L_2$, we integrate (4.3) by parts and obtain for calculating Q_j two equivalent expressions

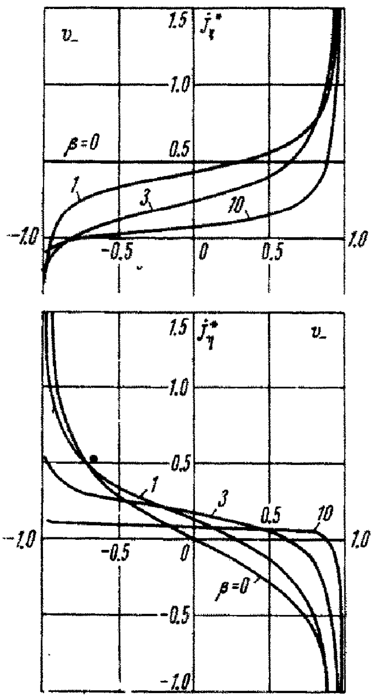

Fig. 2

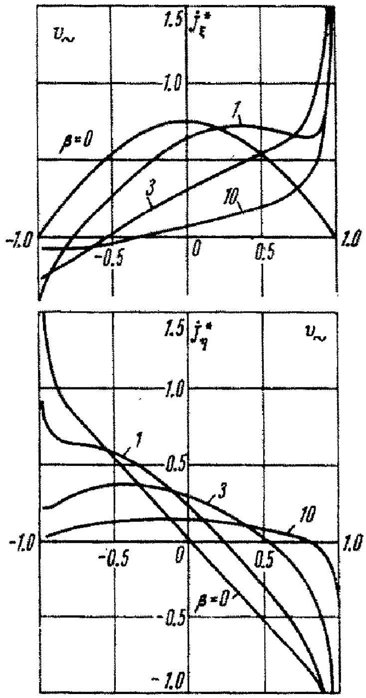

Fig. 3

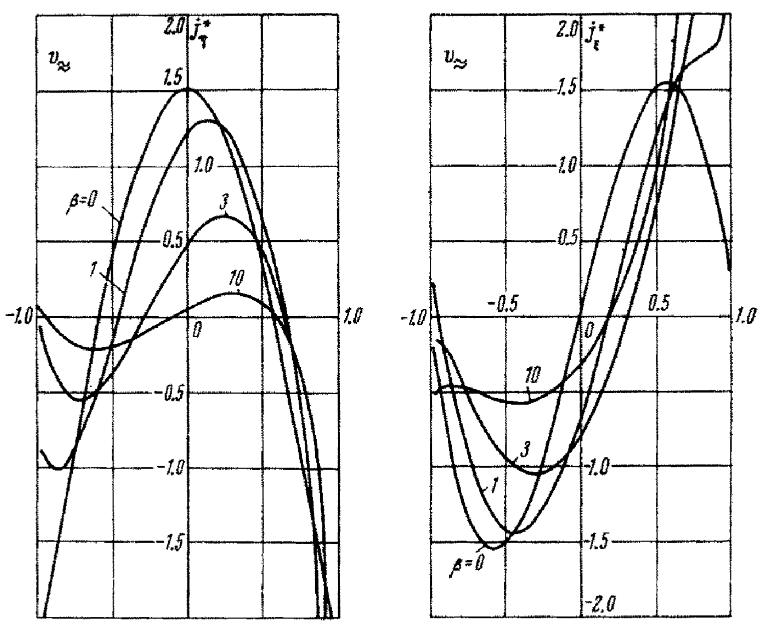

Fig. 4

$$Q_j = -\frac{2H}{c}\int_{-1}^{1}\psi(\xi)\,j_n(\xi)\,d\xi, \qquad \psi(\xi) = \int v(\xi)\,d\xi$$
$$Q_j = \frac{2H}{c}\int_{-1}^{1}\varphi(\xi)\,v(\xi)\,d\xi, \qquad \varphi(\xi) = \int j_n(\xi)\,d\xi \tag{4.6}$$

here $\varphi(\xi)$ is the force function of current and $\psi(\xi)$ the stream function of velocity. For $v(\xi)$ considered here we have

$$\psi_-(\xi) = V\xi, \quad \psi_\sim(\xi) = {}^3/_2V\xi\,(1 - {}^1/_3\xi^2), \quad \psi_\approx(\xi) = 4V\xi^2\,(1 - {}^1/_2\xi^2) \tag{4.7}$$

Using the Cauchy theorem on residues and the Sokhotskii-Plemel formulas, we calculate the integral in the first of formulas (4.6) in its final form [4]. For this it is necessary to take into consideration the expansion of $X_0(\zeta)$ into series

$$X_0(\zeta) = (\zeta+1)^{-1/2+\varepsilon}(\zeta-1)^{-1/2-\varepsilon} = \frac{1}{\zeta}\left(1 + \frac{R_1}{\zeta} + \frac{R_2}{\zeta^2} + \ldots\right) \tag{4.8}$$

where

$$R_1 = 2\varepsilon, \quad R_2 = 2\varepsilon^2 + {}^1/_2, \quad R_3 = {}^4/_3\varepsilon^3 + {}^5/_3\varepsilon, \quad R_4 = {}^2/_3\varepsilon^4 + {}^7/_3\varepsilon^2 + {}^3/_8$$
$$R_5 = {}^4/_{15}\varepsilon^5 + 2\varepsilon^3 + {}^{89}/_{60}\varepsilon, \qquad R_6 = {}^4/_{45}\varepsilon^6 + {}^{11}/_9\varepsilon^4 + {}^{439}/_{180}\varepsilon^2 + {}^5/_{16} \tag{4.9}$$
$$R_7 = {}^8/_{315}\varepsilon^7 + {}^{26}/_{45}\varepsilon^5 + {}^{217}/_{90}\varepsilon^3 + {}^{381}/_{280}$$
$$R_8 = {}^2/_{315}\varepsilon^8 + {}^2/_9\varepsilon^6 + {}^{301}/_{180}\varepsilon^4 + {}^{1247}/_{504}\varepsilon^2 + {}^{35}/_{128}$$

(other coefficients R_μ, $\mu = 9, \ldots$ in expansion (4.8) will not be required in further calculations and formulas).

After several transformations, we obtain for the calculation of the Joule dissipation the following exact analytical expressions:

(a) for $v_- = V = \text{const}$

$$Q_{j-} = 2\pi\,(R_2 - 2\varepsilon R_1) \tag{4.10}$$

(b) for $v_\sim = {}^3/_2V\,(1 - \xi^2)$

$$Q_{j\sim} = {}^8/_{27}\pi\,[(R_6 - 2\varepsilon R_5 + (2\varepsilon^2 - {}^9/_4)\,R_4 + ({}^{25}/_3\varepsilon - {}^4/_3\varepsilon^3)\,R_3 + ({}^9/_2 - 6\varepsilon^2)\,R_2 + (4\varepsilon^3 - 7\varepsilon)\,R_1] \tag{4.11}$$

(c) for $v_\approx = 8V\xi\,(1 - \xi^2)$

$$Q_{j\approx} = {}^1/_8\pi\,[R_8 - 2\varepsilon R_7 + (2\varepsilon^2 - {}^7/_2)\,R_6 + ({}^{19}/_3\varepsilon - {}^4/_3\varepsilon^3)\,R_5 + ({}^2/_3\varepsilon^4 - {}^{17}/_3\varepsilon^2 + {}^{27}/_8)\,R_4 + ({}^8/_3\varepsilon^3 - {}^{14}/_3\varepsilon)\,R_3 + ({}^{10}/_3\varepsilon^2 - {}^4/_3\varepsilon^4 - {}^3/_4)\,R_2] \tag{4.12}$$

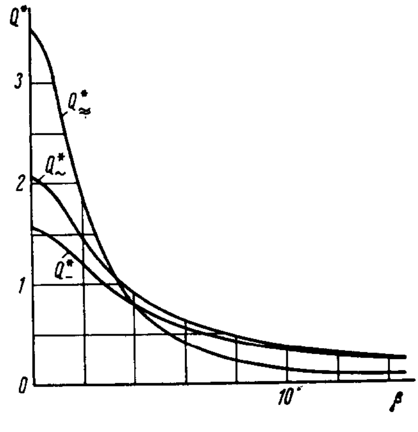

Fig. 5

The dependence of the Joule dissipation Q_{j-}, $Q_{j\sim}$ and $Q_{j\approx}$ on the Hall parameter β calculated by formulas (4.10)–(4.12) and normalized with respect to $\sigma\,(VH\,/\,c)^2$ is shown in Fig. 5. Functions Q_j decrease monotonically with increasing β, and at the limit Q_{j-} and $Q_{j\sim}$ coincide. This is, apparently, a general characteristic of electric fields in this kind of problems: at a fixed flow rate there is always a sufficiently great β beyond which, for any $v(\xi)$ and independently of the velocity profile, Q_j is equal to Q_{j-}, which corresponds to a constant velocity profile $v_- = V = \text{const}$.

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Translated by J. J. D.

DIFFRACTION OF A PLANE WAVE BY A WEDGE MOVING WITH SUPERSONIC SPEED

PMM Vol. 35, №2, 1971, pp. 238-247
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(Received May 15, 1969)

We study the perturbation of the uniform stream behind an oblique shock wave that simultaneously diffracts with an incident wave. The deformation of the shock causes the assignment on its shape of a relation in partial derivatives of the unknown pressure perturbation, which determines the formulation of a Hilbert boundary-value problem for an analytic function.

The classical "problem of diffraction of a plane wave" (by a stationary wedge of finite opening angle), which was solved in 1933 [1], is complicated by assuming that the wedge moves through the gas at supersonic speed.

The problem was briefly considered earlier by the author [2]; an integral of Cauchy type was used to construct its solution. It proves to be convenient here to use the generalization obtained by the author [3] of the solution of a diffraction problem that was constructed in [4, 5]; on it are based the considerations and calculations of the pressure distribution of the wedge surface that are contained in the present paper.

For the special case of a thin wedge moving at hypersonic speed, when Lighthill's solution can be used, the examination was carried out in [6, 7]. Conditions under which interaction is realized without diffraction were indicated in [8]; the analysis performed in [9] was devoted to their small perturbations.

1. Flow field. A wedge of finite opening angle β moves with supersonic speed $W_\infty = M_\infty a_\infty$ in a quiescent ideal gas, forming an attached oblique shock wave that forms an angle α with its symmetry plane. At the instant $t = 0$ it meets the front of a weak plane pressure jump that is propagating through the same gas with a speed a_∞ equal to the speed of sound and making an angle φ with the oblique shock front. The resulting motion is self-similar. The magnitude ε of the pressure jump in the incident wave, referred to the pressure in the quiescent gas, is chosen as the basic small parameter.

Considerations are carried out in the plane perpendicular to the edge of the wedge,